Coding for Euclidean Space: Past, Present and Challenges Ahead

Gottfried Ungerboeck
Broadcom Corporation
• **Past:** from soft-decoding of binary convolutional codes to trellis coded modulation, multilevel coding, dense lattices.

• **Present:** concatenated coding and iterative decoding, joint equalization and decoding, MIMO systems, space-time coding.

• **Outlook:** what comes next – another big topic? extended phase of consolidations? What are the challenges?
Soft / hard decoding (SD/HD) of binary convolutional codes

Heller and Jacobs, 1971 / Lin & Costello, 1st ed, 1983
• Known: soft decoding of binary codes can provide an improvement of $\approx 2$ dB over hard decoding.

• Designing signal codes with large free Euclidean distance could have easily been seen as an important goal.

• But research concentrated on “error control codes”, usually in combination with binary modulation, where Hamming and Euclidean distance are equivalent.

• The paradigm was: hard-decision symbol decoders can make errors; so, transmit redundant check bits and let an error control decoder detect and/or correct the errors.
Receiving signals in additive white Gaussian noise

\[ p(w_n) = \frac{1}{(2\pi)^{N/2} \sigma^N} \exp \left( -\frac{|w_n|^2}{2\sigma^2} \right) \]

N-dim signals

\[ a_n \in A \]

\[ z_n = a_n + w_n \]

ML decoder

\[ \hat{a}_n = \arg \min_{a \in A} |z_n - a|^2 \]

\[ \text{transmitted} \]

\[ a_n \in A \]

\[ a'_n \in A \]

\[ d_{\text{min}}^E \]

\[ d_{\text{min}}^E \]

\[ \text{Pr}[a'_n \text{ more likely than } a_n / z_n] = Q \left( \frac{d_{\text{min}}^E}{2\sigma} \right) \]
Capacity of 2-D AWGN channel with discrete input / soft output

$\log_2(1 + \text{SNR})$

$\text{SNR} = \frac{E_s}{2\sigma^2} [\text{dB}]$

- **64QAM**
- **32QAM**
- **16QAM**
- **8QAM**
- **8PSK**
- **QPSK**
- **BPSK**

Potential coding gain 6-7 dB

GU, 1982
Trellis coded modulation: the first useful code (1975)

4-state coded 8PSK: 3 dB gain over uncoded QPSK (same rate and bandwidth)

Four-state systematic recursive encoder

Set partitioning / mapping

S.P. Mapping

Four-state trellis diagram
From binary modulation to higher order modulations

Spectral efficiency [bit/sec/Hz]

Free Euclidean distance relative to QPSK [dB]

Binary convolutional codes with QPSK modulation (4-256 states)

Trellis coded modulation (4-256 states)

Uncoded modulation

BPSK

QPSK

8PSK

16QAMsq

32QAMcr

128QAMcr

64QAMsq

128QAMsq

64QAMsq

8PSK

32QAMcr

8QAMds

8PSK

16QAMsq

8QAMds

8PSK

R=1/3

R=1/2

R=3/4
TCM: Effective coding gain vs. decoding complexity

- Effective coding gain (dB)
- Normalized complexity

- Ung 1D
- Ung 2D
- Wei 4D
- Wei 8D
- 8-state, 2D (V.32)
- 16-state
- 32-state, 2D
- 32-state, 4D
- 64-state, 8D
- 64-state
Applications of TCM

• Dial-up modems: V.32, V.17, V.34, V.90, V.92
• Digital subscriber links: SHDLS, ADSL, VDSL
• Cable modems: downstream J.83, upstream Docsis 2.0
• Terrestrial TV: VSB modem
• Ethernet: 802.3 1 Gbit/s over copper
• Wireless LAN: 802.11g
• Mobile telephony: GSM Edge
• WPAN: 802.15  ... etc.
Multilevel coding and multistage decoding (Imai&Hirakawa 1977)

(Lattice = infinite symbol constellation with algebraic group properties)

\( \mathbb{Z}^2 : \Delta_0^2 \)

\( y^0 \leftarrow R\mathbb{Z}^2 : \Delta_1^2 = 2\Delta_0^2 \)

\( y^1 \leftarrow 2\mathbb{Z}^2 : \Delta_2^2 = 4\Delta_0^2 \)

\( 2R\mathbb{Z}^2 : \Delta_3^2 = 8\Delta_0^2 \)

Multi-level construction: 
Gosset lattice \( E_8 \) from \((\mathbb{Z}^2)^4\)

<p>| | | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>0</td>
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<td>1</td>
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<td>2</td>
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</table>

\( \mathbb{Z}^2 \) \( \mathbb{Z}^2 \) \( \mathbb{Z}^2 \) \( \mathbb{Z}^2 \)

\( (d_{\min}^E)^2 = \min [\Delta_0^2 \times 4, \Delta_1^2 \times 2, \Delta_2^2 \times 1] = 4\Delta_0^2 \)

Coding gain \( E_8 \) over \( \mathbb{Z}^8 \): 
\[ \gamma_c = \frac{(d_{\min}^E)^2}{\Delta_0^2} \times \left( \frac{2^{1+3+4}}{2^{12}} \right)^2 = 2 \text{ (3 dB)} \]

Multistage decoding (= suboptimal bounded distance decoding)

1. Find most-likely codeword \( \hat{c}^0 \in \mathcal{C}^0 \), assuming unconstrained \( c^1, c^2 \)
2. Find most-likely codeword \( \hat{c}^1 \in \mathcal{C}^1 \) for given \( c^0 = \hat{c}^0 \), assuming unconstrained \( c^2 \)
3. Find most-likely codeword \( \hat{c}^2 \in \mathcal{C}^2 \) for given \( c^0 = \hat{c}^0 \) and \( c^1 = \hat{c}^1 \).
**Dense lattices**

**Barnes-Wall Lattices**

<table>
<thead>
<tr>
<th>Schläfi</th>
<th>Gosset</th>
<th>(\gamma_c)</th>
<th>(K_{\min})</th>
<th>(K_{\min} / N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Z^2)</td>
<td>(D_4)</td>
<td>1 (0 dB)</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(E_8)</td>
<td>(\sqrt{2}) (1.5 dB)</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(\Lambda_{16})</td>
<td>2 (3 dB)</td>
<td>240</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>(\Lambda_{32})</td>
<td>(\sqrt{2}) (1.5 dB)</td>
<td>4'320</td>
<td>270</td>
</tr>
<tr>
<td></td>
<td>(\Lambda_{64})</td>
<td>4 (6 dB)</td>
<td>146'880</td>
<td>4590</td>
</tr>
<tr>
<td></td>
<td>(\Lambda_{128})</td>
<td>(4\sqrt{2}) (7.5 dB)</td>
<td>9'694'080</td>
<td>151'470</td>
</tr>
</tbody>
</table>

Constructions are based on set partitioning of component integer lattices and multilevel coding with Reed-Muller codes (many equivalent constructions)

**Hexagonal**

- \(\gamma_c = 1.15\) (0.62 dB)
- \(K_{\min} = 6\)
- \(K_{\min} / N = 3\)

**Leech**

- \(\gamma_c = 4\) (6 dB)
- \(K_{\min} = 196'560\)
- \(K_{\min} / N = 8190\)
Symbol error probability per 2-D \( \approx K_{\text{min}}/(N/2) \times Q\left(\sqrt{\gamma_c \text{ snr}}\right) \)
Types of code concatenation

Parallel concatenation

Serial concatenation

Product codes

Low density parity check codes (Gallager 1963)

"Checks on checks": row and column codes must be linear and over the same field
Cycle-free Tanner graph and Pearl’s “belief” propagation

Variable node with noisy observation

Variable node w/o observation ("hidden" variable)

Check node

Subtree $T_2$

Subtree $T_3$

Subtree $T_1$

$a$ priori probability

likelihood from observation

$a$ posteriori probability

check-to-variable likelihoods = extrinsic probabilities

variable-to-check probabilities
Parallel concatenated codes, turbo decoding (1993)

Iterative ("Turbo") decoding introduced by Berrou & Glavieux achieves near-optimum decoding by message passing of marginal probabilities. This leads to cycles in the Tanner graph. The effect of cycles is mitigated by using long random-like interleaving.

- optimum decoding (no cycles)
- requires passing of joint probabilities
- too complex, except for very short codes
Iterative decoding of parallel concatenated code (PCC)

Tanner graph explaining Turbo decoding

For every i \((1 \leq i \leq K)\):

**extrinsic** marginal probability from decoder 1

is used as **a priori** probability for decoder 2

\[
\begin{align*}
\Pi^{-1} & \quad \Pi \\
\lambda^{(n)}_{E_iU}(u_i)'s & \quad \Pi_U^{(N)}(u_i)'s
\end{align*}
\]
Cycles, states, and forward-backward algorithm

1. Cycles can be eliminated by introducing states.
2. States are composite hidden variables or functions of variables.

Tanner graph without states: cycles

\[ y_n = y_{n-1} \oplus y_{n-2} \oplus y_{n-1} \]

Tanner-Wiberg-Loeliger graph: states eliminate cycles

Belief propagation = forward-backward = BCJR algorithm
• **Benedetto, Divsalar, Montorsi, Pollara** (1996-98): estimated BER for *ML decoding* of PCCCs and SCCCs based on *average weight* enumeration assuming *uniform random interleaving*. Results illustrate interleaver gain, *near Shannon-limit performance* at moderate SNR, *change of slope* due to *low-weight codewords* at high SNR.

• **Gallager** (1963), **Richardson and Urbanke** (2001): *density evolution* in message-passing decoding for ensemble of random infinite-length LDPC codes; *SNR threshold for convergence* interpreted as “capacity” of LDPC code.

Iterative equalization and decoding

\[ x_n = \sum_{\ell=0}^{L} h_\ell a_{n-\ell} + w_n \]

- **Trellis based APP equalizer** - too complex (except in very simple cases)

- **Linear MMSE equalizer with a priori info** (symbol means and variances)
  - no a priori info: ordinary MMSE equalizer
  - perfect a priori info: ISI cancellation + matched filtering

Iterative equalization and decoding: EXtrinsic Information Transfer chart

- Mutual information $I(X, \text{Decoder out}) = I(X, \text{Equalizer in})$ [bit]
- Mutual information $I(X, \text{Equalizer out}) = I(X, \text{Decoder in})$ [bit]

Approx. linear MMSE
SISO decoding
Trellis-based APP/EP
Exact linear MMSE

BCC $G=[7,5]$, binary modulation, Proakis C channel, $E_s/N_0 = 4$ dB
(Tuechler et al., May 2002)
Multiple-input multiple-output (MIMO) channels

Singular value decomposition \( H = U_{NT} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} V_{NR} \)

Rank \( M \leq \text{Min}(N_T, N_R) \)

Fixed \( H \): Capacity \( C(H) = \sum_{i=1}^{M} \log_2 (1 + \rho \lambda_i) \left( = \log_2 \left[ \det \left( I + \rho HH^* \right) \right] \right) \), \( \rho = \frac{E_s}{N_T} \frac{1}{N_0} \)

Random \( H \): Ergodic capacity \( \bar{C} = \text{average } C(H) \); Outage capacity \( C_p : \Pr(C(H) < C_p) = p \, (\%) \)
Frequency selective channels can be converted into non-frequency selective vector channels by FDM (OFDM/DMT, filter banks)

### Spatial MIMO equalization

<table>
<thead>
<tr>
<th>Linear at receiver</th>
<th>right-multiply H by $H^{-1}(r)$</th>
</tr>
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<tbody>
<tr>
<td>Linear at transmitter</td>
<td>left-multiply H by $H^{-1}(l)$</td>
</tr>
<tr>
<td>Linear at transmitter and receiver</td>
<td>left-multiply H by $U^<em>$, right-multiply H by $V^</em>$ (SVD)</td>
</tr>
<tr>
<td>Nonlinear at receiver</td>
<td>matrix DFE (BLAST, MUD)</td>
</tr>
<tr>
<td>Nonlinear at transmitter</td>
<td>matrix precoding (MUP)</td>
</tr>
</tbody>
</table>

(for spatial signals in row-vector form)
MIMO system: decentralized transmitters, centralized receiver

Spatial DFE = simple* multi-user detection (MUD) = BLAST
* advanced MUD employs iterative processing

\[ \mathbf{a} \mathbf{H} + \mathbf{n} = \mathbf{y} \]

Performance depends on order of detection

\[
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    0 & r_{22} & r_{23} \\
    0 & 0 & r_{33}
\end{bmatrix}
\begin{bmatrix}
    r_{11}^{-1} & 0 & 0 \\
    0 & r_{22}^{-1} & 0 \\
    0 & 0 & r_{33}^{-1}
\end{bmatrix}
= \begin{bmatrix}
    1 & b_{12} & b_{13} \\
    0 & 1 & b_{23} \\
    0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\begin{bmatrix}
a_1 & a_2 & a_3
\end{bmatrix}
= \begin{bmatrix}
a_1 a_{12} + a_2 & a_1 b_{13} + a_2 b_{23} + a_3
\end{bmatrix}
\]

QR decompositions \( \mathbf{H} = \mathbf{R}_1 \mathbf{Q}_1 = \mathbf{Q}_2 \mathbf{R}_2 \ldots \) \( \mathbf{R} = \) upper or lower triangular, \( \mathbf{Q} = \) unitary; different orderings...

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Spatial TH precoding = Multi-user precoding (MUP)

\[
\begin{align*}
\mathbf{x}_1 &= a_1 \\
\mathbf{x}_2 &= (a_2 - x_1 b_{12}) + k_2 M_p \\
\mathbf{x}_3 &= (a_3 - x_1 b_{13} - x_2 b_{23}) + k_3 M_p
\end{align*}
\]

Receiver diversity

M independently fading paths

Maximimum ratio combining

\[ E\{|w|^2\} = N_0 (|h_1|^2 + \cdots + |h_M|^2) \]

\[ y = \left( |h_1|^2 + \cdots + |h_M|^2 \right) \sqrt{E_s} a + w \]

M-branch diversity gain

Source: H. Boelcskei
Transmit diversity and a simple space-time code

Alamouti (IEEE JSAC, Nov 98)

\[
\begin{bmatrix}
  a_1 & a_2 \\
  -a_2^* & a_1^*
\end{bmatrix}
\]

\[
h_1
\]

\[
h_2
\]

\[
[x_1 \ x_2] = [h_1 \ h_2] \begin{bmatrix}
  a_1 & a_2 \\
  -a_2^* & a_1^*
\end{bmatrix} + [n_1 \ n_2]
\]

\[
\begin{bmatrix}
  h_1^* & h_2^* \\
  x_2^* & -x_1^*
\end{bmatrix}
\]

\[
[y_1 \ y_2] = (|h_1|^2 + |h_2|^2) [a_1 \ a_2] + [n_1^* \ n_2^*]
\]

2-branch diversity gain
MIMO channels and space-time coding: objectives

- **Spatial multiplexing gain**
  - creation of multiple channels within same bandwidth

- **Diversity gain**
  - mitigation of fading losses by averaging over individually fading paths

- **Coding gain**
  - increasing distance between codewords / sequences by improved arrangements of points in higher dimensional spaces
Throughput in MIMO-OFDM cellular systems

Source: H. Boelcskei
What comes next in coding?

- Error control coding
- Signal-space coding
- Concat. codes and turbo decoding
- MIMO and S-T coding

Timeline:
- 1948: Error control coding
- 1978: Signal-space coding
- 1993: Concat. codes and turbo decoding
- 1998: MIMO and S-T coding
- 2002: Question mark

(a) That's all, folks!
(b) Another big topic?
(c) Extended phase of consolidations?
Challenges ahead

- Develop new standards exploiting the exciting potential of MIMO and S/T coding: G3+ cellular systems, Gbit/s wireless Ethernet, … using base stations with multiple antennas, mobile stations with 1 - 2 antennas.
  - optimize network capacity by S/T coding, multi-user detection, multi-user precoding, beam forming, …
  - channel estimation; PHY and MAC protocols

- Performance / complexity in turbo and S/T coding: many schemes proposed, few adopted in applications, what is most useful?
Challenges ahead

• Concatenated codes & iterative decoding for very low error rates: e.g., 10 Gbit/s optical links, BER=$10^{-12} - 10^{-15}$.

• Soft-decoding of widely used algebraic FEC codes: known soft-decoding algorithms are too complex or provide only small gain, for e.g., RS(255,239).

• Designing chips of massive complexity

• Develop market for these new technologies. Succeed under slower market conditions.